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Solution of a problem on non-negative subset sums

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ABSTRACT

Let n and r be positive integers with $1 \leq r \leq n-1$. Solving a problem of Chiaselotti–Marino–Nardi, which is a generalization of a problem of Manickam and Miklós, we prove that for each integer q with $2^{n-1}+1 \leq q \leq 2^n - 2^{n-r} + 1$ there exists an n-tuple $(a_1, \ldots, a_n)$ of integers such that $\sum_{i=1}^n a_i \geq 0$, $a_1, \ldots, a_r \geq 0$, $a_{r+1}, \ldots, a_n < 0$ and there are exactly q subsets X of $\{1, \ldots, n\}$ with $\sum_{i \in X} a_i \geq 0$.

1. Introduction

Let $a_1, \ldots, a_r$ be r non-negative real numbers and let $a_{r+1}, \ldots, a_n$ be $n-r$ negative real numbers with non-negative sum, i.e.,

$$\sum_{i=1}^{r} a_i + \sum_{j=r+1}^{n} a_j \geq 0.$$

Let $\gamma(n,r)[\eta(n,r)]$ be the minimum [maximum] number of subsets of $\{a_1, \ldots, a_n\}$ whose element-sum is non-negative. In [7], the authors have studied the following two problems:

(P1) *Which are the values of* $\gamma(n,r)$ *and* $\eta(n,r)$ *for each* n *and* r, $1 \leq r \leq n-1$?

(P2) *If* q *is an integer such that* $\gamma(n,r) \leq q \leq \eta(n,r)$, *can we find* r *non-negative real numbers* $a_1, \ldots, a_r$ *and* $n-r$ *negative real numbers* $a_{r+1}, \ldots, a_n$ with $\sum_{i=1}^{r} a_i + \sum_{j=r+1}^{n} a_j \geq 0$ *such that the number of subsets of* $\{a_1, \ldots, a_n\}$ *whose element-sum is non-negative is exactly* q?

Using results of [3], Chiaselotti et al. solved the problem (P1) and they provided a partial result for (P2) showing that the answer is affirmative for a related, but not equivalent (or sufficient) poset-formulation. In this paper we solve completely problem (P2).

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Our study is closely related to the well-known Manickam–Miklós–Singhi-conjecture: In 1987 and 1988 Manickam et al. [10,11] stated the following conjecture: If $d \in \mathbb{N}$ and $n \geq 4d$, there exist at least $\binom{n-1}{d-1}$ d-element subsets of $\{a_1, \ldots, a_n\}$ whose element-sum is non-negative.

This conjecture was treated in [1,2,5,6,9,12,14] and solved for different values of the parameters involved. Bisi and Chiaselotti [4] have discovered an interesting lattice structure hidden behind this conjecture. It turns out that this lattice is isomorphic to the poset $M(r) \times M(n-r)^*$ which was shown to be a Peck poset by Stanley [13] in order to solve the Erdős–Moser problem. For more details see [8, pp. 14 and 243].

2. The result

Let n be a positive integer and let $[n] = \{1, 2, \ldots, n\}$. For two integers k and ℓ with $k \leq \ell$ let $[k, \ell] = \{k, k+1, \ldots, \ell\}$. Let r be an integer in $[n-1]$. Let

$$A_{n,r} = \left\{ (a_1, \ldots, a_n) : \sum_{i=1}^{n} a_i \geq 0, a_1, \ldots, a_r \geq 0, a_{r+1}, \ldots, a_n < 0 \right\}.$$

For $\mathbf{a} = (a_1, \ldots, a_n) \in A_{n,r}$ and $X \subseteq [n]$ let $\Sigma_{\mathbf{a}}(X) = \sum_{i \in X} a_i$,

$$\mathcal{F}_{\mathbf{a}}^{+} = \{X \subseteq [n] : \Sigma_{\mathbf{a}}(X) \geq 0\} \quad \text{and} \quad \mathcal{F}_{\mathbf{a}}^{-} = \{X \subseteq [n] : \Sigma_{\mathbf{a}}(X) < 0\}.$$

Because always $\mathbf{a}$ will be clear from the context we omit in the following the index $\mathbf{a}$.

Note that $\emptyset \in \mathcal{F}^+$, $[n] \in \mathcal{F}^+$ and $|\mathcal{F}^+| + |\mathcal{F}^-| = 2^n$. Since $X \in \mathcal{F}^-$ implies $[n] \setminus X \in \mathcal{F}^+$ we obtain $|\mathcal{F}^+| \geq 2^{n-1} + 1$. Since $X \subseteq [r+1, n]$ and $X \neq \emptyset$ imply $X \in \mathcal{F}^-$ we have $|\mathcal{F}^+| \leq 2^n - 2^{n-r} + 1$.

Solving a problem of Chiaselotti et al. [7] which is a generalization of a problem of Manickam and Miklós [10] we prove the following theorem:

Theorem 1. *Let n and r be positive integers with $1 \leq r \leq n-1$ and let q be an integer with $2^{n-1} + 1 \leq q \leq 2^n - 2^{n-r} + 1$. Then there exists an n-tuple $\mathbf{a} \in A_{n,r}$ such that $|\mathcal{F}^+| = q$.*

Proof. For $X \subseteq [n]$ we set $X^+ = X \cap [r]$ and $X^- = X \cap [r+1, n]$.

Case 1. $2^{n-1} + 1 \leq q \leq 2^n - 2^{n-1} + 2^{r-1}$.

Then we define $\mathbf{a}$ as follows:

$$a_i = \begin{cases} q - 2^r & \text{if } i = 1, \\ 2^{r-i} & \text{if } i \in [2, r], \\ -2^{i-2} & \text{if } i \in [r+1, n]. \end{cases}$$

Then $a_1 \geq 0$ and $\sum_{i=1}^{n} a_i = q - 2^r + (2^{r-1} - 1) - 2^{r-1}(2^{n-r} - 1) = q - 1 - 2^{n-1} \geq 0$, hence indeed $\mathbf{a} \in A_{n,r}$. Let

$$\begin{aligned} \mathcal{F}_1 &= \{X \subseteq [n] : X^- = \emptyset\}, \\ \mathcal{F}_2 &= \{X \subseteq [n] : X^- \neq \emptyset, 1 \notin X\}, \\ \mathcal{F}_3 &= \{X \subseteq [n] : X^- \neq \emptyset, 1 \in X, X^- \neq [r+1, n]\}, \\ \mathcal{F}_4 &= \{X \subseteq [n] : 1 \in X, X^- = [r+1, n]\}. \end{aligned}$$

Obviously,

$$\{X : X \subseteq [n]\} = \mathcal{F}_1 \dot\cup \mathcal{F}_2 \dot\cup \mathcal{F}_3 \dot\cup \mathcal{F}_4.$$

It is clear that $\mathcal{F}_1 \subseteq \mathcal{F}^+$.

Now let $X \in \mathcal{F}_2$. Then $\Sigma(X) = \Sigma(X^+) + \Sigma(X^-) \leq 2^{r-1} - 1 - 2^{r-1} < 0$, hence $\mathcal{F}_2 \subseteq \mathcal{F}^-$.

If $X \in \mathcal{F}_3$ then $\Sigma(X) = \Sigma(X^+) + \Sigma(X^-) \geq q - 2^r - 2^{r-1}(2^{n-r} - 2) \geq 2^{n-1} + 1 - 2^r - 2^{r-1}(2^{n-r} - 2) = 1 \geq 0$ hence $\mathcal{F}_3 \subseteq \mathcal{F}^+$.

Finally let $X \in \mathcal{F}_4$. We put $Y = X^+ \setminus \{1\}$. Then $\Sigma(X) = q - 2^r + \Sigma(Y) - (2^{n-1} - 2^{r-1})$. Hence $\Sigma(X) \geq 0$ iff $\Sigma(Y) \geq 2^{n-1} + 2^{r-1} - q$.

In the actual Case 1 we have $0 \le 2^{n-1} + 2^{r-1} - q \le 2^{r-1} - 1$. Using binary expansions it is easy to see that $\Sigma(Y)$ takes on all integer values from 0 to $2^{r-1} - 1$ exactly once if Y runs through all subsets of $[2, r]$. Hence,

$$|\mathcal{F}_4 \cap \mathcal{F}^+| = 2^{r-1} - 1 - (2^{n-1} + 2^{r-1} - q) + 1 = q - 2^{n-1}.$$

Consequently, we have

$$|\mathcal{F}^+| = |\mathcal{F}_1| + |\mathcal{F}_3| + |\mathcal{F}_4 \cap \mathcal{F}^+| = 2^r + 2^{r-1}(2^{n-r} - 2) + (q - 2^{n-1}) = q.$$

Case 2. $2^n - 2^{n-1} + 2^{r-1} < q \le 2^n - 2^{n-r} + 1$.

Then there exists a unique integer $j \in [2, r]$ such that

$$2^n - 2^{n-j+1} + 2^{r-j+1} < q \le 2^n - 2^{n-j} + 2^{r-j}. \tag{1}$$

Let

$$d = q - 2^n + 2^{n-j+1} - 2^{r-j+1}.$$

By (1),

$$0 < d \le 2^{n-j} - 2^{r-j}. \tag{2}$$

We write d in the form

$$d = k \cdot 2^{r-j} + \ell,$$

where k is a non-negative integer and $1 \le \ell \le 2^{r-j}$, i.e.

$$k = \left\lceil \frac{d}{2^{r-j}} \right\rceil - 1, \quad \ell = d - k \cdot 2^{r-j}.$$

Note that by (2)

$$0 \le k \le 2^{n-r} - 2.$$

We define $\mathbf{a}$ as follows:

$$a_i = \begin{cases} 2^{n-1} - 2^{r-1} & \text{if } i \in [j-1], \\ (k+1) \cdot 2^{r-1} - 2^{r-j} + \ell & \text{if } i = j, \\ 2^{r-i} & \text{if } i \in [j+1, r], \\ -2^{i-2} & \text{if } i \in [r+1, n]. \end{cases}$$

Then $a_i \ge 0$ for $i \in [j]$ and $\sum_{i=1}^n a_i \ge 2^{n-1} - 2^{r-1} - 2^{r-1}(2^{n-r} - 1) = 0$, hence indeed $\mathbf{a} \in A_{n,r}$. Let

$$\begin{aligned} \mathcal{F}_1 &= \{X \subseteq [n] : X^+ \cap [j-1] \ne \emptyset\}, \\ \mathcal{F}_2 &= \{X \subseteq [n] : X^+ \cap [j-1] = \emptyset, j \notin X^+, X^- \ne \emptyset\}, \\ \mathcal{F}_3 &= \{X \subseteq [n] : X^+ \cap [j-1] = \emptyset, j \notin X^+, X^- = \emptyset\}, \\ \mathcal{F}_4 &= \{X \subseteq [n] : X^+ \cap [j-1] = \emptyset, j \in X^+\}. \end{aligned}$$

Obviously,

$$\{X : X \subseteq [n]\} = \mathcal{F}_1 \dot\cup \mathcal{F}_2 \dot\cup \mathcal{F}_3 \dot\cup \mathcal{F}_4.$$

If $X \in \mathcal{F}_1$ then $\Sigma(X) \ge (2^{n-1} - 2^{r-1}) - 2^{r-1}(2^{n-r} - 1) = 0$, hence $\mathcal{F}_1 \subseteq \mathcal{F}^+$.

Now let $X \in \mathcal{F}_2$. Then $\Sigma(X) = \Sigma(X^+) + \Sigma(X^-) \le (2^{r-j} - 1) - 2^{r-1} < 0$, hence $\mathcal{F}_2 \subseteq \mathcal{F}^-$.

Clearly, $\mathcal{F}_3 \subseteq \mathcal{F}^+$.

Now we study $\mathcal{F}_4$. Let

$$\begin{aligned} \mathcal{F}_{4,1} &= \{X \in \mathcal{F}_4 : \Sigma(X^-) \ge -k \cdot 2^{r-1}\}, \\ \mathcal{F}_{4,2} &= \{X \in \mathcal{F}_4 : \Sigma(X^-) \le -(k+2) \cdot 2^{r-1}\}, \\ \mathcal{F}_{4,3} &= \{X \in \mathcal{F}_4 : \Sigma(X^-) = -(k+1) \cdot 2^{r-1}\}. \end{aligned}$$

Then

$$\mathcal{F}_4 = \mathcal{F}_{4,1} \dot{\cup} \mathcal{F}_{4,2} \dot{\cup} \mathcal{F}_{4,3}.$$

If $X \in \mathcal{F}_{4,1}$ then $\Sigma(X) > ((k+1)2^{r-1} - 2^{r-j}) - k \cdot 2^{r-1} = 2^{r-1} - 2^{r-j} \geq 0$, hence $\mathcal{F}_{4,1} \subseteq \mathcal{F}^+$. Obviously, $\Sigma(X^-)$ takes on all values from $\{0, -2^{r-1}, -2 \cdot 2^{r-1}, \ldots, -(2^{n-r} - 1) \cdot 2^{r-1}\}$ exactly once if X^- runs through all subsets of $[r+1, n]$. Hence

$$|\mathcal{F}_{4,1}| = (k+1) \cdot 2^{r-j}.$$

If $X \in \mathcal{F}_{4,2}$ then $\Sigma(X) \leq ((k+1) \cdot 2^{r-1} - 2^{r-j} + \ell) + (2^{r-j} - 1) - (k+2) \cdot 2^{r-1} = \ell - 1 - 2^{r-1} \leq 2^{r-j} - 1 - 2^{r-1} < 0$, hence $\mathcal{F}_{4,2} \subseteq \mathcal{F}^-$.

Finally, let $X \in \mathcal{F}_{4,3}$. We put $Y = X^+ \setminus \{j\}$. Then $\Sigma(X) = ((k+1) \cdot 2^{r-1} - 2^{r-j} + \ell) + \Sigma(Y) - (k+1) \cdot 2^{r-1}$, hence $\Sigma(X) \geq 0$ iff $\Sigma(Y) \geq 2^{r-j} - \ell$. Obviously, $\Sigma(Y)$ takes on all integer values from 0 to $2^{r-j} - 1$ exactly once if Y runs through all subsets of $[j+1, r]$. Accordingly,

$$|\mathcal{F}_{4,3} \cap \mathcal{F}^+| = \ell.$$

Consequently,

$$\begin{aligned} |\mathcal{F}^+| &= |\mathcal{F}_1| + |\mathcal{F}_3| + |\mathcal{F}_{4,1}| + |\mathcal{F}_{4,3} \cap \mathcal{F}^+| \\ &= (2^n - 2^{n-j+1}) + 2^{r-j} + (k+1) \cdot 2^{r-j} + \ell \\ &= (2^n - 2^{n-j+1}) + 2^{r-j+1} + d \\ &= q. \quad \square \end{aligned}$$

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